J Stat Phys (2007) 128: 1153–1158
DOI 10.1007/s10955-007-9356-3

Condensation for a Fixed Number of Independent Random Variables

Pablo A. Ferrari · Claudio Landim · Valentin V. Sisko

Received: 29 December 2006 / Accepted: 24 May 2007 /
Published online: 29 June 2007

Abstract A family of m independent identically distributed random variables indexed by a chemical potential $\varphi \in [0, \gamma]$ represents piles of particles. As φ increases to γ, the mean number of particles per site converges to a maximal density $\rho_c < \infty$. The distribution of particles conditioned on the total number of particles equal to n does not depend on φ (canonical ensemble). For fixed m, as n goes to infinity the canonical ensemble measure behave as follows: removing the site with the maximal number of particles, the distribution of particles in the remaining sites converges to the grand canonical measure with density ρ_c; the remaining particles concentrate (condensate) on a single site.

Keywords Condensation · Critical density

1 Introduction

Condensation phenomena appears in many physical systems. From a physical point of view, in Bose-Einstein condensation a large fraction of the atoms collapses into the lowest quantum state, which is possible to observe macroscopically. From a mathematical point of view

P.A. Ferrari (✉)
IME USP, Caixa Postal 66281, 05315-970 Sao Paulo, Brazil
e-mail: pablo@ime.usp.br

C. Landim · V.V. Sisko
IMPA, Estrada Dona Castorina 110, CEP 22460-320 Rio de Janeiro, Brazil

C. Landim
e-mail: landim@impa.br

V.V. Sisko
e-mail: valentin@impa.br

C. Landim
CNRS UMR 6085, Université de Rouen, UMR 6085, Avenue de l'Université, BP.12, Technopôle du Madrillet, 76801 Saint-Etienne-du-Rouvray, France

the condensation phenomena can be interpreted as the spontaneous migration of a macroscopic number of particles to some region of the space. A physical account of different models and discussion may be found in the recent articles: Evans [1], Jeon and March [8], Jeon, March and Pittel [9], Godrèche [5], Grosskinsky, Schütz and Spohn [7], Evans, Majumdar and Zia [3, 4], Evans and Hanney [2], Majumdar, Evans and Zia [11] and Godrèche and Luck [6].

These references consider the following mathematical model. Fix a natural number m. The state space consists of particle configurations $\xi \in \mathbb{N}^m$; ξ_i represents the number of particles at site $i \in \{1, \dots, m\}$. Consider a family μ_φ of product measures on $\mathbb{N}^m$ indexed by a chemical potential φ with range in $[0, \gamma]$ for some $\gamma > 0$. The density $\rho = R(\varphi)$ (mean number of particles per site) is an increasing function of φ. We assume that the maximal density is finite: $\rho_c = R(\gamma) < \infty$.

For an integer $n \geq 0$, denote by ν_n the canonical measure associated to the family μ_φ. This is the measure μ_φ conditioned on the hyperplane of configurations with n particles: $\nu_n(\xi) = \mu_\varphi(\xi \mid \sum_i \xi_i = n)$. In this context, condensation means that all but a few particles concentrate on the same site. We prove in this article such a statement under some conditions on the product measure μ_φ. More precisely, we show that if we remove the site with the largest number of particles, the projection of the canonical measure on the remaining sites converges to the product measure with maximal density ρ_c, when the total number of sites m is fixed and the total number n of particles increases to infinity.

2 Notation and Result

Denote by $\mathbb{N}$ the nonnegative integers and let $f : \mathbb{N} \to \mathbb{R}_+$ be a positive real function such that

$$\lim_{n\to\infty} \frac{f(n)}{f(n+1)} = \gamma, \qquad \sum_{n\geq 0} \gamma^n f(n) < \infty. \tag{1}$$

Fix a natural number m and assume there exists a positive constant $C(m)$ such that

$$\frac{\max_{n/m\leq k\leq n} \gamma^k f(k)}{\min_{n/m\leq k\leq n} \gamma^k f(k)} \leq C(m) \tag{2}$$

for all $n \geq 1$.

Remark 1 If $\gamma^n f(n)$ is decreasing and there exists a positive constant $C(m)$ such that

$$\gamma^{n/m} f(n/m) \leq C(m)\gamma^n f(n) \tag{3}$$

for all n, then f satisfies (2). For instance

$$f(n) = n^{-\alpha}$$

with $\alpha > 1$ satisfies (1) and (3) (and hence (2)) for $\gamma = 1$ and $C(m) = m^\alpha$.

We consider a family of product measures in the state space $\mathbb{N}^m$. Configurations in this space are denoted by the Greek letter $\xi = (\xi_1, \dots, \xi_m)$, with ξ_j in $\mathbb{N}$ for $1 \leq j \leq m$. Let

$Z:\mathbb{R}_+\to\mathbb{R}_+$ the partition function defined by

$$Z(\varphi)=\sum_{n\geq 0}\varphi^n f(n).$$

It follows from (1) that the radius of convergence of Z is equal to γ: $Z(\varphi)<\infty$ if and only if $\varphi\leq\gamma$. For $0\leq\varphi\leq\gamma$, denote by $\mu_{m,\varphi}$ the grand canonical measure on $\mathbb{N}^m$ given by

$$\mu_{m,\varphi}(\xi)=\frac{1}{Z(\varphi)^m}\prod_{j=1}^m\varphi^{\xi_j}f(\xi_j).$$

Denote by $|\xi|$ the total number of particles of the configuration: $|\xi|=\xi_1+\cdots+\xi_m$ and by $\Sigma_{m,n}$, $n\geq 1$, the subspace of configurations of $\mathbb{N}^m$ with n particles:

$$\Sigma_{m,n}=\{\xi\in\mathbb{N}^m:|\xi|=n\}. \tag{4}$$

For $n\geq 0$, let $\nu_{m,n}$ be the canonical measure concentrated on configurations with n particles:

$$\nu_{m,n}(\xi)=\mu_{m,\varphi}(\xi\,||\xi|=n).$$

Notice that the right hand side does not depend on the parameter φ.

Denote by $\mathcal{X}_m$ the space of monotone configurations:

$$\mathcal{X}_m=\{\zeta\in\mathbb{N}^m:\zeta(1)\leq\cdots\leq\zeta(m)\}.$$

Configurations of $\mathcal{X}_m$ are denoted by the Greek letter ζ. Define the order operator $\mathsf{O}:\mathbb{N}^m\to\mathcal{X}_m$ which takes a configuration of $\mathbb{N}^m$ to the monotone configuration by reordering appropriately the coordinates:

$$(\mathsf{O}\xi)_i\leq(\mathsf{O}\xi)_j\quad\text{for } 1\leq i\leq j\leq m,$$
$$(\mathsf{O}\xi)_i=\xi_{\sigma(i)}\quad\text{for a permutation }\sigma\text{ of the set }\{1,\ldots,m\}.$$

Define the cut operator $\mathsf{C}:\mathbb{N}^m\to\mathbb{N}^{m-1}$ which eliminates the last coordinate:

$$(\mathsf{C}\xi)_i=\xi_i\quad\text{for } 1\leq i\leq m-1.$$

These operators induce two measures: $\hat\mu_{m,\varphi}$ defined on $\mathcal{X}_m$ and $\hat\nu_{m-1,n}$ defined on $\mathcal{X}_{m-1}$:

$$\hat\mu_{m,\varphi}=\mu_{m,\varphi}\mathsf{O}^{-1},\qquad \hat\nu_{m-1,n}=\nu_{m,n}(\mathsf{C}\circ\mathsf{O})^{-1}.$$

To obtain a configuration with law $\hat\mu_{m,\varphi}$, sample a configuration with the product measure $\mu_{m,\varphi}$ and order it. To obtain a configuration with law $\hat\nu_{m-1,n}$, sample a configuration with the product measure $\mu_{m,\varphi}$ conditioned to have n particles (i.e. $\nu_{m,n}$), order it and cut the last coordinate.

Next we state our result. It asserts that if the total number of sites m is fixed and the total number n of particles increases to infinity, the canonical measure concentrated on configurations with n particles, from which we removed the site with the largest number of particles, converges to the ordered *maximal* grand canonical measure on the remaining sites. Maximal means that we set φ to be γ.

By symmetry the position of the maximum site is uniformly distributed in $\{1, \dots, m\}$. Sampling from the ordered measure $\hat{\nu}_{m-1,n}$ and reordering randomly and uniformly the labels of the particles we obtain a measure which approaches the maximal product measure in $n-1$ remaining sites as n increases.

Theorem 1 *Assume that f is a nonincreasing function satisfying hypotheses (1) and (2). For each $m \geq 2$, the sequence of probability measures $\hat{\nu}_{m-1,n}$ converges weakly to $\hat{\mu}_{m-1,\gamma}$ as n tends to infinity.*

Remark 2 A similar result has been proven in various of the quoted references in the thermodynamical limit, as the total number of sites diverges, as well as the total number of particles. See for instance [7], where the number of particles n divided the number of sites $m = m(n)$ converges to a constant $\rho > \rho_c$ as n goes to infinity. Theorem 1 shows that this phenomenon is a combinatorial fact that can be observed without making the number of sites grow to infinity.

Remark 3 Relation with the zero range process. Fix $m \geq 1$, an irreducible symmetric transition probability $p(x, y)$ on $\{1, \dots, m\}$ and a positive function $g : \mathbb{N} \to \mathbb{R}_+$. The zero-range process associated to (p, g) can be informally described as follows. Particles evolve on $\{1, \dots, m\}$. If there are k particles at some site x of $\{1, \dots, m\}$ one of them jumps to site y at rate $g(k)p(x, y)$, independently from what happens at the other sites.

Let $f(k)^{-1} = g(1) \cdots g(k)$ and assume that f satisfies assumption (1) for some $\gamma > 0$. It is well known [10] that the product measures $\{\mu_{m,\varphi} : 0 \leq \varphi \leq \gamma\}$ defined above are invariant for the symmetric zero-range process.

Let $R^* = \sum_{n \geq 0} n\gamma^n f(n)$ and assume that $R^* < \infty$. The function $R : [0, \gamma] \to [0, R^*]$ defined by $R(\varphi) = \mu_{m,\varphi}[\xi_1]$ gives the density of particles under the invariant measure $\mu_{m,\varphi}$. A simple computation of R' shows that R is bijective. Since in the symmetric case any product invariant measure belongs to the set $\{\mu_{m,\varphi} : 0 \leq \varphi \leq \gamma\}$, there are no product invariant measures with density above R^*.

Since the process is ergodic, for every $n \geq 1$, there exists a unique stationary measure, denoted above by $\nu_{m,n}$. Order the configuration and remove the site with the largest number of particles. Under the additional assumption (2), Theorem 1 states that this measure converges, as $n \uparrow \infty$, to the ordered maximal product state. In particular, all but a finite number of particles tend to accumulate in one site.

3 Proof of Theorem 1

For an ordered configuration ζ belonging to $\mathcal{X}_m$ (resp. η belonging to $\mathcal{X}_{m-1}$), let $K_m(\zeta)$ (resp. $K_{m-1,n}(\eta)$) be the number of configurations in $\mathbb{N}^m$ whose ordering (resp. ordering and cutting) gives ζ (resp. η):

$$K_m(\zeta) = \sharp\{\xi \in \mathbb{N}^m : \mathsf{O}(\xi) = \zeta\},$$

$$K_{m-1,n}(\eta) = \sharp\{\xi \in \Sigma_{m,n} : \mathsf{C} \circ \mathsf{O}(\xi) = \eta\}.$$

Note that the number of elements of an empty set is zero. Since $\mu_{m,\varphi}$, $\nu_{m,n}$ are invariant by permutation, for any ζ in $\mathcal{X}_m$, η in $\mathcal{X}_{m-1}$,

$$\begin{aligned} \hat{\mu}_{m,\varphi}(\zeta) &= K_m(\zeta)\mu_{m,\varphi}(\zeta), \\ \hat{\nu}_{m-1,n}(\eta) &= K_{m-1,n}(\eta)\nu_{m,n}(\eta_1, \dots, \eta_{m-1}, n - |\eta|). \end{aligned} \tag{5}$$

Lemma 1 *For any η in $\mathcal{X}_{m-1}$ and any $n \geq 1$, $K_{m-1,n}(\eta) \leq m K_{m-1}(\eta)$. Moreover, for any η in $\mathcal{X}_{m-1}$,*

$$\lim_{n\to\infty} K_{m-1,n}(\eta) = m K_{m-1}(\eta).$$

The proof is elementary. A fixed ordered configuration $(\eta_1, \ldots, \eta_{m-1})$ belonging to $\mathcal{X}_{m-1}$ arises from a permutation of the coordinates η_j. To compute $K_{m-1,n}$, we add the coordinate $n - |\eta|$. This extra coordinate accounts for the factor m which corresponds to all its possible positions. If n is large this last coordinate is different from all others and provides m distinct vectors.

We are now in position to prove Theorem 1. Fix η in $\mathcal{X}_{m-1}$. In view of (5),

$$\hat{\nu}_{m-1,n}(\eta) = \frac{K_{m-1,n}(\eta) f(\eta_1)\cdots f(\eta_{m-1}) f(n-|\eta|)}{\sum_{\eta'\in\mathcal{X}_{m-1}} K_{m-1,n}(\eta') f(\eta'_1)\cdots f(\eta'_{m-1}) f(n-|\eta'|)}.$$

By (5) and Lemma 1, to prove Theorem 1, we only need to show that

$$\begin{aligned}\lim_{n\to\infty} &\sum_{\eta'\in\mathcal{X}_{m-1}} K_{m-1,n}(\eta') f(\eta'_1)\cdots f(\eta'_{m-1}) \frac{f(n-|\eta'|)}{f(n-|\eta|)} \\ &= \sum_{\eta'\in\mathcal{X}_{m-1}} m K_{m-1}(\eta') \gamma^{|\eta'|-|\eta|} f(\eta'_1)\cdots f(\eta'_{m-1}). \end{aligned} \tag{6}$$

Fix a positive constant $M > 1$. By assumption (1) and by Lemma 1, for configurations such that $|\eta| \leq M$, $\lim_n K_{m-1,n}(\eta) = m K_{m-1}(\eta)$ and

$$\lim_{n\to\infty} f(n-|\eta|)/f(n-|\eta'|) = \gamma^{|\eta'|-|\eta|}.$$

In particular, for every $M \geq 1$,

$$\begin{aligned}\lim_{n\to\infty} &\sum_{\substack{\eta'\in\mathcal{X}_{m-1}\\ |\eta'|\leq M}} K_{m-1,n}(\eta') f(\eta'_1)\cdots f(\eta'_{m-1}) \frac{f(n-|\eta'|)}{f(n-|\eta|)} \\ &= \sum_{\substack{\eta'\in\mathcal{X}_{m-1}\\ |\eta'|\leq M}} m K_{m-1}(\eta') \gamma^{|\eta'|-|\eta|} f(\eta'_1)\cdots f(\eta'_{m-1}). \end{aligned}$$

To estimate the sum $|\eta'| > M$, recall from Lemma 1 that

$$K_{m-1,n}(\eta') \leq m K_{m-1}(\eta').$$

On the other hand, since

$$\eta_m = n - |\eta| \geq \max_{1\leq j\leq m-1} \eta_j, \qquad \eta'_m = n - |\eta'| \geq \max_{1\leq j\leq m-1} \eta'_j,$$

we have that $n/m \leq \eta_m$, $\eta'_m \leq n$. Thus, by assumption (2),

$$\frac{f(n-|\eta'|)}{f(n-|\eta|)} \leq C_1(m) \gamma^{|\eta'|-|\eta|}.$$

Therefore,

$$\sum_{\substack{\eta'\in\mathcal{X}_{m-1}\\|\eta'|>M}} K_{m-1,n}(\eta')f(\eta'_1)\cdots f(\eta'_{m-1})\frac{f(n-|\eta'|)}{f(n-|\eta|)}$$
$$\le C(m)\sum_{\substack{\eta'\in\mathcal{X}_{m-1}\\|\eta'|>M}} K_{m-1}(\eta')\gamma^{|\eta'|-|\eta|}f(\eta'_1)\cdots f(\eta'_{m-1}).$$

This expression vanishes as $M\uparrow\infty$ because the sum without the constraint $|\eta'|>M$ is finite, being equal to $\gamma^{-|\eta|}Z(\gamma)^{m-1}$, where $Z(\gamma)$ is the partition function introduced in Sect. 2. This concludes the proof of the theorem.

Acknowledgements The authors are grateful to CNPq, FAPERJ, FAPESP, and PRONEX for financial support.

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